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A new approach for solving mixed boundary value problems along holes in orthotropic plates

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Abstract

A new analytical method for solving mixed boundary value problems along holes in composite plates is presented. This addresses problems, where a part of the hole boundary is stress-free, and the other part is subjected to displacement or/and load boundary conditions. The present approach simplifies and speeds up the numerical calculations for the implementation of the boundary conditions by deriving stress functions, which automatically satisfy the stress-free boundary condition over a part of the hole boundary. Only the boundary conditions on the loaded part of the hole need, therefore, to be enforced, typically by a numerical technique such as collocation. Application of this method to orthotropic laminates with a pin-loaded hole problem is discussed. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Boundary value; Orthotropic plates; Load boundary conditions

1. Introduction

Mixed boundary value problems in solid mechanics are intrinsically difficult to solve analytically owing to the mixed nature of the boundary conditions: stresses are prescribed on part of the boundary, and displacements on the remaining part. The mathematical difficulties encountered in the analytical derivation are supplemented by numerical difficulties found when enforcing the boundary conditions. A specific type of mixed boundary value problem is that of plates (isotropic or anisotropic), which contain one or more holes on the boundary on which load or displacements are applied. This work presents a general theory for anisotropic plates with a hole, where a part of the hole boundary is stress-free. A wide range of practical problems fall into this category. A typical example is the pin-loaded hole problem. In this case, the region of the hole that is not in contact with the pin is the stress-free part of the boundary.

The present theory is able to simplify the problem by incorporating the stress-free boundary conditions directly into the expression for the stress functions, from which the stresses and displacements in the plate are directly derived. The stresses and displacements therefore, automatically satisfy the stress-free boundary conditions, and only the remaining boundary conditions on the rest of the hole boundary need to be

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enforced. This simplifies the nature of the problem by simplifying the boundary conditions, and can even reduce it to a simple boundary value problem if the remaining boundary conditions on the rest of the hole boundary are pure displacement conditions. In addition, the numerical methods used to enforce these remaining boundary conditions can be implemented faster, hence cutting down computing costs, because the length of the hole boundary involved is shorter.

The present approach is based on the theory of anisotropic elasticity developed by Lekhnitskii (1968) and subsequently by Savin (1961). A brief review of the theory applied to anisotropic plates containing holes is presented in Section 2.1. In order to reduce the mathematical calculations, the case of orthotropic plate containing a single hole will be treated here, but the theory can be easily extended to the case of an anisotropic plate containing multiple holes. The derivation of stress functions satisfying the stress-free boundary conditions on part of the hole boundary is given in Section 2.2.

2. Theory

2.1. Complex variable theory

The present work is based on the theory of anisotropic elasticity developed by Lekhnitskii (1968). This approach leads to an analytical expression for the stress functions $\{\phi_k(z_k)\}_{k=1,2}$ in the form of series with unknown coefficients. The stresses and displacements in the plate can be calculated, once these stress functions are determined. Eq. (1) give the relation between the displacements (u, v) in the $\vec{x}$ and $\vec{y}$ directions, respectively, and the stress functions at a point $M(x,y)$ on the plane:

$$\begin{cases} u(x,y) = 2\,\Re\mathrm{e}\{p_1\,\phi_1(z_1) + p_2\,\phi_2(z_2)\}, \\ v(x,y) = 2\,\Re\mathrm{e}\{q_1\,\phi_1(z_1) + q_2\,\phi_2(z_2)\}, \end{cases} \tag{1}$$

with

$$z_k = x + \mu_k y \quad \text{for } k = 1,2. \tag{2}$$

The coefficients μ_k, p_k and q_k depend only on the plate elasticity constants and are given in Appendix A. The in-plane stresses ($\sigma_x, \sigma_y, \tau_{xy}$) are related to the stress functions by

$$\begin{cases} \sigma_x(x,y) = 2\,\Re\mathrm{e}\left\{\mu_1^2\,\phi_1'(z_1) + \mu_2^2\,\phi_2'(z_2)\right\}, \\ \sigma_y(x,y) = 2\,\Re\mathrm{e}\left\{\phi_1'(z_1) + \phi_2'(z_2)\right\}, \\ \tau_{xy}(x,y) = -2\,\Re\mathrm{e}\left\{\mu_1\phi_1'(z_1) + \mu_2\,\phi_2'(z_2)\right\}, \end{cases} \tag{3}$$

where a prime (′) indicates differentiation with respect to the variable z_1 or z_2.

The unknown coefficients in the stress functions $\{\phi_k\}_{k=1,2}$ are determined by applying the boundary conditions on the hole surface that correspond to the physical problem. These boundary conditions are by nature of a mixed-mode type, with displacement conditions enforced over part of the hole, and stress conditions enforced over the rest of the hole boundary.

Lekhnitskii (1968) and Savin (1961) showed that the stress functions for an anisotropic plate containing a hole and subjected to a uniform stress state at infinity, $\{\phi_k(z_k)\}_{k=1,2}$ are of the form:

$$\phi_k(z_k) = A_k \log(z_k) + \left(B_k^* + \mathrm{i}C_k^*\right) z_k + \phi_k^0(z_k) \quad \text{for } k = 1,2, \tag{4}$$

where $\{\phi_k^0(z_k)\}_{k=1,2}$ are holomorphic functions in the region S outside the hole, and go to zero at infinity. They are therefore of the form:

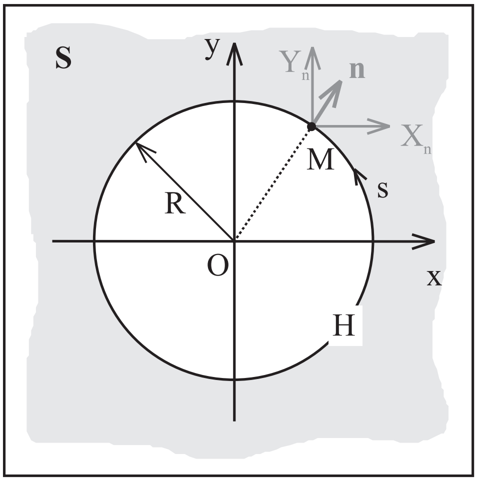

Fig. 1. Loads on the hole boundary.

$$\phi_k^0(z_k) = \sum_{n=0}^{\infty} \frac{b_n^{(k)}}{z_k^n} \quad \text{for } k = 1, 2. \tag{5}$$

In Eq. (4), the logarithm represents the presence of the hole. The coefficients B_k^* and C_k^* represent the contribution from the uniform stress state at infinity and can be related to these remote stresses (Savin, 1961). For simplicity, it is considered here that no remote stresses are applied, and therefore $B_k^* = C_k^* = 0$.

In order to express the boundary conditions on the hole, the stress functions $\phi_k^0(z_k)$ must be related to the forces $X_n(s)$ and $Y_n(s)$ in the $\vec{x}$ and $\vec{y}$ directions, respectively, at a point M located on the hole edge, Fig. 1. The real s is the coordinate of the point $M(x,y)$ along the hole boundary. From Savin (1961)

$$\begin{cases} 2\,\Re\mathrm{e}\{\phi_1(z_1) + \phi_2(z_2)\} = -\int_0^s Y_n\,\mathrm{d}s + C_1 = f_1(x,y), \\ 2\,\Re\mathrm{e}\{\mu_1\phi_1(z_1) + \mu_2\phi_2(z_2)\} = \int_0^s X_n\,\mathrm{d}s + C_2 = f_2(x,y), \end{cases} \tag{6}$$

where the complex numbers z_k are related to the coordinates (x, y) by Eq. (2), and C_k are constants. Eq. (6) define the functions $f_1(x,y)$ and $f_2(x,y)$ as equal to the right-hand sides.

From Eq. (6), an expression for the resultant vector $(X_{AB} + \mathrm{i}Y_{AB})$ acting on an arc (AB) of the hole boundary can be derived:

$$\begin{aligned} X_{AB} + \mathrm{i}Y_{AB} &= \int_A^B X_n + \mathrm{i}Y_n\,\mathrm{d}s \\ \Rightarrow\ X_{AB} + \mathrm{i}Y_{AB} &= \left[(\mu_1 - \mathrm{i})\phi_1(z_1) + \left(\bar{\mu}_1 - \mathrm{i}\right)\bar{\phi}_1(z_1) + (\mu_2 - \mathrm{i})\phi_2(z_2) + \left(\bar{\mu}_2 - \mathrm{i}\right)\bar{\phi}_2(z_2)\right]_A^B, \end{aligned} \tag{7}$$

where a top bar ($\bar{\ }$) represents the complex conjugate.

The coefficients A_k in Eq. (4) are then related to the resultant vector $(X + \mathrm{i}Y)$ of forces on the hole boundary by expressing first, the condition that $(X + \mathrm{i}Y)$ is equal to the quantity $(X_n(s) + \mathrm{i}Y_n(s))$ integrated over the whole hole boundary H, and second, that displacements $(u(s), v(s))$ on the hole do not increase in value after a complete turn around H, Fig. 1. This leads to the following pair of equations:

$$\begin{cases} \int_H X_n(s) + \mathrm{i}Y_n(s)\,\mathrm{d}s = X + \mathrm{i}Y, \\ \int_H u(s) + \mathrm{i}v(s)\,\mathrm{d}s = 0. \end{cases} \tag{8}$$

Substituting Eqs. (1) and (7) (with $A = B$) into Eq. (8), the following system of equations is obtained:

$$\begin{cases} (1+\mathrm{i}\mu_1)A_1 - \left(1+\mathrm{i}\bar{\mu}_1\right)\bar{A}_1 + (1+\mathrm{i}\mu_2)A_2 - \left(1+\mathrm{i}\bar{\mu}_2\right)\bar{A}_2 = -\frac{X+\mathrm{i}Y}{2\pi}, \\ (p_1+\mathrm{i}q_1)A_1 - \left(\bar{p}_1+\mathrm{i}\bar{q}_1\right)\bar{A}_1 + (p_2+\mathrm{i}q_2)A_2 - \left(\bar{p}_2+\mathrm{i}\bar{q}_2\right)\bar{A}_2 = 0. \end{cases} \tag{9}$$

In Eq. (9), we have used the fact that around H, the function $\{\log(z)\}$ undergoes an increase, equal to $(2\pi i)$. The orthotropy of the problem ensures that the vector resultant of forces on H is along the $\vec{x}$ axis. Therefore, $X = F$ and $Y = 0$. Solving Eq. (9) yields the complex coefficients A_1 and A_2 under the following form:

$$A_k = F\,\Delta_k \quad \text{for } k = 1, 2. \tag{10}$$

The complex constants $\{\Delta_k\}_{k=1,2}$ are given in Appendix A.

In the following calculations, it is more convenient to perform a conformal transformation and map the outside of the hole S of radius R into the inside of the unit circle (γ) (Fig. 2). This conformal transformation is defined by Eq. (11) (Savin, 1961):

$$z_k(\zeta) = \frac{R}{2}\left[(1+\mathrm{i}\mu_k)\zeta + \frac{1-\mathrm{i}\mu_k}{\zeta}\right] \iff \zeta(z_k) = \frac{z_k - \sqrt{z_k^2 - R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}. \tag{11}$$

Using Eq. (11), the stress functions $\{\phi_k(z_k)\}_{k=1,2}$ in Eq. (4) can now be expressed as a function of ζ, the coordinate image of z_k by the conformal transformation,

$$\phi_k(z_k) = \Phi_k(\zeta) = -F\,\Delta_k \log(\zeta) + \Phi_k^0(\zeta) \quad \text{for } k = 1, 2. \tag{12}$$

It can be proved (Berbinau and Soutis, 1998) that the new functions $\{\Phi_k^0(\zeta)\}_{k=1,2}$ defined by Eq. (12) are holomorphic inside the unit circle. Eq. (6) then becomes

$$\begin{cases} 2\,\Re\{\Phi_1(\sigma) + \Phi_2(\sigma)\} = f_1(\theta), \\ 2\,\Re\{\mu_1\Phi_1(\sigma) + \mu_2\Phi_2(\sigma)\} = f_2(\theta). \end{cases} \tag{13}$$

Here the argument of the functions $\{\Phi_k(\zeta)\}_{k=1,2}$ is $\sigma = \mathrm{e}^{\mathrm{i}\theta}$ as the points are on the unit circle (Fig. 2). Now, let us define functions $\{f_k^0(\theta)\}_{k=1,2}$ as follows:

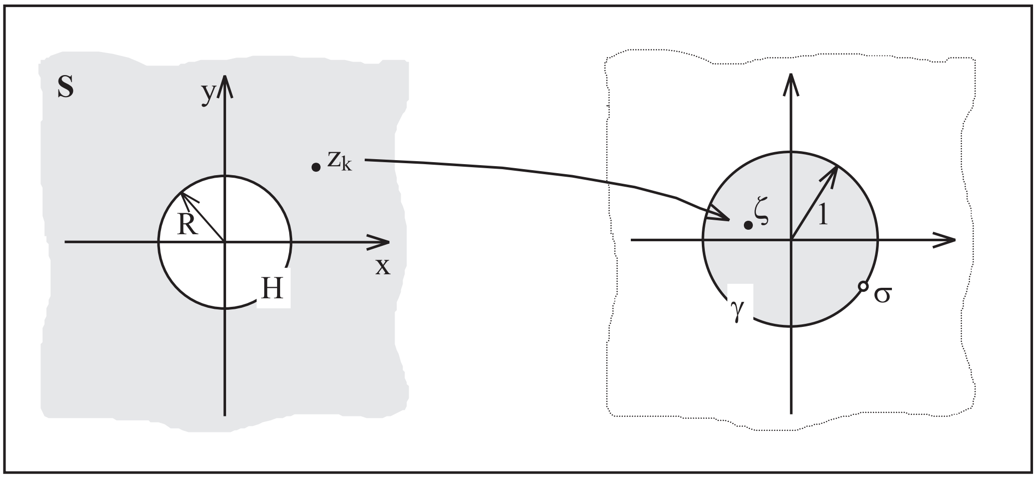

Fig. 2. Conformal transformation.

$$\begin{cases} 2\,\Re\mathrm{e}\left\{\Phi_1^0(\sigma)+\Phi_2^0(\sigma)\right\}=f_1^0(\theta), \\ 2\,\Re\mathrm{e}\left\{\mu_1\Phi_1^0(\sigma)+\mu_2\Phi_2^0(\sigma)\right\}=f_2^0(\theta). \end{cases} \tag{14}$$

The $f_k^0(\theta)$ functions can be related to the $f_k(\theta)$ functions using Eqs. (12) and (13), i.e.

$$\begin{cases} f_1^0(\theta)=f_1(\theta)+2F\,\Re\mathrm{e}\{(\Delta_1+\Delta_2)\log(\sigma)\}, \\ f_2^0(\theta)=f_2(\theta)+2F\,\Re\mathrm{e}\{(\mu_1\Delta_1+\mu_2\Delta_2)\log(\sigma)\}. \end{cases} \tag{15}$$

The cornerstone of the present theoretical approach is the Schwartz formula mentioned by Muskhelishvili (1963) and Savin (1961). The Schwartz formula relates a function $F(\zeta)$ holomorphic inside the unit circle (γ) to the value of its real part $\{\Re\mathrm{e}[F]\}(\theta)$ on the contour of the unit circle and is given by

$$F(\zeta)=\frac{1}{2\pi\mathrm{i}}\oint_\gamma\{\Re\mathrm{e}[F]\}(\theta)\frac{\sigma+\zeta}{\sigma-\zeta}\frac{\mathrm{d}\sigma}{\sigma}+\text{constant}. \tag{16}$$

Applying the Schwartz formula to the holomorphic functions $\Phi_k^0(\zeta)$ and using Eq. (14), one obtains

$$\begin{cases} \Phi_1^0(\zeta)=\frac{\mathrm{i}}{4\pi(\mu_1-\mu_2)}\oint_\gamma\left[\mu_2 f_1^0(\theta)-f_2^0(\theta)\right]\frac{\sigma+\zeta}{\sigma-\zeta}\frac{\mathrm{d}\sigma}{\sigma}, \\ \Phi_2^0(\zeta)=\frac{-\mathrm{i}}{4\pi(\mu_1-\mu_2)}\oint_\gamma\left[\mu_1 f_1^0(\theta)-f_2^0(\theta)\right]\frac{\sigma+\zeta}{\sigma-\zeta}\frac{\mathrm{d}\sigma}{\sigma}. \end{cases} \tag{17}$$

The constants in Eq. (16) have been discarded as they have no influence on the stresses.

The boundary value problem is now ready to be solved. Indeed, the functions $f_k(\theta)$ can be calculated using Eq. (6) from X_n and Y_n, which are known directly from the stress boundary conditions around the hole. The functions $f_k^0(\theta)$ can then be calculated using Eq. (15). Substituting $f_k^0(\theta)$ into Eq. (17) and evaluating the integrals, the functions $\Phi_k^0(\zeta)$ are obtained. The functions $\Phi_k(\zeta)$ can then be derived from Eq. (12). Using the conformal transformation, Eq. (11) will ultimately give the stress functions $\phi_k(z_k)$ from which stresses and displacements around the hole can be calculated from Eqs. (3) and (1), respectively.

2.2. Case of a hole with no stresses on part of its boundary

In all previous analytical work on problems with boundary conditions applied to a hole boundary (see for instance De Jong, 1982; Murthy et al., 1991; Naidu et al., 1985; Mangalgiri and Dattaguru, 1986; Ireman et al., 1993; Rangavittal, 1995), the Schwartz formula was not used. Instead, the stress functions given by Eq. (12) were written in the following form, using the fact that the functions $\Phi_k^0(\zeta)$ are holomorphic inside the unit circle:

$$\phi_k(z_k)=\Phi_k(\zeta)=-F\Delta_k\log(\zeta)+\sum_{n=1}^{\infty}a_n^{(k)}\zeta^n \quad \text{for } k=1,2, \tag{18}$$

where the coefficients $a_n^{(k)}$ are complex constants. The boundary conditions on the hole (including the stress-free conditions if applicable) were then enforced in order to calculate the associated coefficients $a_n^{(k)}$ by an appropriate numerical method, such as the collocation method.

In the present work, the aim is to incorporate a part of the boundary conditions (namely the stress-free conditions) directly into the stress functions, so that these stress functions satisfy automatically the stress-free conditions. The numerical procedure for the enforcement of the remaining boundary conditions will then be simplified, and less time consuming. Such an approach is possible as the holomorphic stress functions $\Phi_k^0(\zeta)$ have been written in the form of Eq. (17), thanks to the use of the Schwartz formula (16). The derivation of stress functions that automatically satisfy the stress-free boundary conditions on part of the hole is performed below.

A part of the hole boundary is therefore, assumed to be stress-free, and the rest of the hole boundary is subjected to displacement conditions, stress conditions, or a mixture of both. As the plate is orthotropic

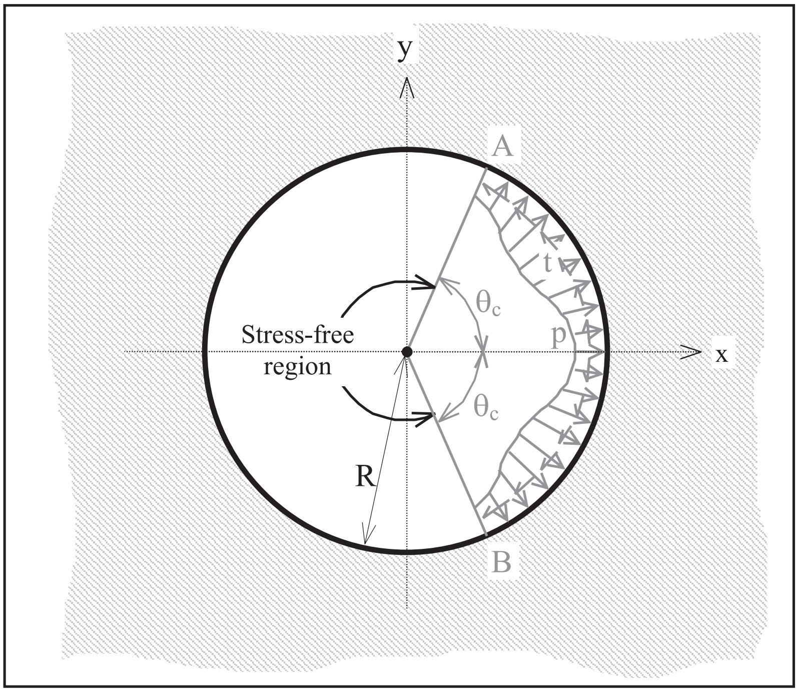

Fig. 3. Geometry of a partially loaded hole in an orthotropic plate.

(the loading axis $\vec{x}$ is along one of the principal directions), the boundary conditions are symmetrical with respect to the $\vec{x}$ axis. Hence, the geometry of the problem is as indicated in Fig. 3. The hole radius is R. The plate is considered infinite.

The angle θ_c defines the stress-free region and the region, where the boundary conditions are applied. The stress-free region is thus defined by $|\theta| > \theta_c$. The region, where boundary conditions are applied is defined by $|\theta| \leqslant \theta_c$. In the latter region, the boundary conditions will be combinations of a radial and a circumferential boundary condition chosen among the following four conditions:

$$\text{radial conditions} \quad \begin{cases} u_r(R,\theta) = D_r(\theta) \\ \sigma_r(R,\theta) = S_r(\theta) \end{cases} \quad \text{for } |\theta| \leqslant \theta_c, \tag{19a}$$

$$\text{circumferential conditions} \begin{cases} u_\theta(R,\theta) = D_\theta(\theta) \\ \tau_{r\theta}(R,\theta) = S_{r\theta}(\theta) \end{cases} \quad \text{for } |\theta| \leqslant \theta_c. \tag{19b}$$

In Eqs. (19a) and (19b), $u_r(r,\theta)$ and $u_\theta(r,\theta)$ are the radial and circumferential components of the displacements, respectively. Similarly, $\sigma_r(r,\theta)$ and $\tau_{r\theta}(r,\theta)$ are the normal and shear components of the stresses, respectively. The functions $D_r(\theta)$, $S_r(\theta)$, $D_\theta(\theta)$, and $S_{r\theta}(\theta)$ are known functions of the angle θ and represent the boundary conditions along the hole.

The loaded region defined by $|\theta| \leqslant \theta_c$ may be subjected, in the most general case, to a normal pressure $p(\theta)$ and shear $t(\theta)$ distributions, Fig. 3. The X_n and Y_n components of the force at a point $M(s)$, Fig. 1, are therefore (Lekhnitskii, 1968; Berbinau and Soutis, 1998),

$$\begin{cases} X_n = -p(\theta)\frac{dy}{ds} + t(\theta)\frac{dx}{ds}, \\ Y_n = p(\theta)\frac{dx}{ds} + t(\theta)\frac{dy}{ds}. \end{cases} \tag{20}$$

The function f_1 and f_2 defined by Eq. (7) can now be written in terms of p and t by substituting X_n and Y_n from Eq. (20)

$$\begin{cases} f_1(\theta) = R\int p(\theta)\sin(\theta)\,\mathrm{d}\theta + R\int t(\theta)\cos(\theta)\,\mathrm{d}\theta + k_1, \\ f_2(\theta) = R\int p(\theta)\cos(\theta)\,\mathrm{d}\theta - R\int t(\theta)\sin(\theta)\,\mathrm{d}\theta + k_2. \end{cases} \quad \text{where } k = \text{constant.} \tag{21}$$

Following Savin (1961), the unloaded region on the hole boundary starts at the point A defined by $\theta = \theta_c$. Hence at point A, $f_1 = f_2 = 0$. This gives the values of the two constants in Eq. (21), and eventually the following expression for $f_1(\theta)$ and $f_2(\theta)$ are obtained:

$$\begin{cases} f_1(\theta) = R\left(\int p(\theta)\sin(\theta) + t(\theta)\cos(\theta)\,\mathrm{d}\theta\right) - R\left\{\left[\int p(\theta)\sin(\theta) + t(\theta)\cos(\theta)\,\mathrm{d}\theta\right]_{\theta_c}\right\} \\ f_2(\theta) = R\left(\int p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta\right) - R\left\{\left[\int p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta\right]_{\theta_c}\right\} \end{cases} \quad \text{for } |\theta| \leqslant \theta_c. \tag{22}$$

Eq. (22) are valid for $\theta_c \geqslant \theta \geqslant -\theta_c$. Indeed, when θ reaches $(-\theta_c)$, the first integrals in $f_1(\theta)$ and $f_2(\theta)$ become equal to their value at $\theta = -\theta_c$. Beyond $(-\theta_c)$, i.e. in the region $|\theta| \geqslant \theta_c$, they remain constant as that region is stress-free. Therefore, the functions $f_1(\theta)$ and $f_2(\theta)$ are constant over $|\theta| > \theta_c$ and equal to

$$\begin{cases} f_1(\theta) = R\left\{\left[\int p(\theta)\sin(\theta) + t(\theta)\cos(\theta)\,\mathrm{d}\theta\right]_{-\theta_c}\right\} - R\left\{\left[\int p(\theta)\sin(\theta) + t(\theta)\cos(\theta)\,\mathrm{d}\theta\right]_{\theta_c}\right\} \\ f_2(\theta) = R\left\{\left[\int p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta\right]_{-\theta_c}\right\} - R\left\{\left[\int p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta\right]_{\theta_c}\right\} \end{cases}$$
$$\text{for } |\theta| > \theta_c \tag{23}$$

Now, as the boundary conditions are symmetrical with respect to the $\vec{x}$ axis, $p(\theta)$ is even and $t(\theta)$ is odd. Thus, Eq. (23) becomes

$$\begin{cases} f_1(\theta) = 0 \\ f_2(\theta) = -2R\left\{\left[\int p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta\right]_{\theta_c}\right\} \end{cases} \quad \text{for } |\theta| > \theta_c. \tag{24}$$

Now, by moving once around the hole from point A in the clockwise direction, functions $f_1(\theta)$ and $f_2(\theta)$ will increase by amounts equal to the right-hand sides in Eq. (24). Hence, from Eq. (6), the resultant of the forces on the hole boundary are

$$\begin{cases} X = -2R\left\{\left[\int p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta\right]_{\theta_c}\right\} \\ Y = 0. \end{cases} \tag{25}$$

There is no component of the resulting force in the $\vec{y}$ direction. This was to be expected as the problem is symmetrical with respect to the $\vec{x}$ axis, and therefore, the resultant F of the loads on the hole boundary is in the $\vec{x}$ direction. The resultant force F is therefore,

$$F = -2R\int_0^{\theta_c} p(\theta)\cos(\theta) - t(\theta)\sin(\theta)\,\mathrm{d}\theta. \tag{26}$$

Now, the functions f_1 and f_2 need to be expressed in terms of σ before we can calculate the integrals in Eq. (17). Using $\sigma = \mathrm{e}^{\mathrm{i}\theta}$ in Eqs. (23) and (25), we obtain

$$f_1(\sigma) = \begin{cases} -\frac{R}{2}\int\left(1-\frac{1}{\sigma^2}\right)p(\sigma) + \mathrm{i}\left(1+\frac{1}{\sigma^2}\right)t(\sigma)\,\mathrm{d}\sigma \\ \quad +\frac{R}{2}\left[\int\left(1-\frac{1}{\sigma^2}\right)p(\sigma) + \mathrm{i}\left(1+\frac{1}{\sigma^2}\right)t(\sigma)\,\mathrm{d}\sigma\right]_{\theta_c} \quad \text{for } |\theta| \leqslant \theta_c \\ 0 \quad \text{for } |\theta| > \theta_c, \end{cases} \tag{27}$$

$$f_2(\sigma)=\begin{cases}\frac{R}{2}\int -\mathrm{i}\left(1+\frac{1}{\sigma^2}\right)p(\sigma)+\left(1-\frac{1}{\sigma^2}\right)t(\sigma)\,\mathrm{d}\sigma \\ \quad -\frac{R}{2}\left[\int -\mathrm{i}\left(1+\frac{1}{\sigma^2}\right)p(\sigma)+\left(1-\frac{1}{\sigma^2}\right)t(\sigma)\,\mathrm{d}\sigma\right]_{\theta_c} & \text{for } |\theta|\leqslant\theta_c \\ -R\left[\int -\mathrm{i}\left(1+\frac{1}{\sigma^2}\right)p(\sigma)+\left(1-\frac{1}{\sigma^2}\right)t(\sigma)\,\mathrm{d}\sigma\right]_{\theta_c} & \text{for } |\theta|>\theta_c.\end{cases} \tag{28}$$

In Eqs. (27) and (28), the functions $p(\sigma)$ and $t(\sigma)$ are defined from the previous functions $p(\theta)$ and $t(\theta)$ by the expressions $p[\theta(\sigma)]$ and $t[\theta(\sigma)]$. Substituting Eq. (15) into Eq. (17), after simplifications, we obtain for $k=1,2$

$$\Phi_k^0(\zeta)=\frac{\mathrm{i}(-1)^{k+1}}{4\pi(\mu_1-\mu_2)}\left\{\frac{-\mathrm{i}F}{2\pi}\oint_\gamma\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{\log(\sigma)}{\sigma}\,\mathrm{d}\sigma+\oint_\gamma\left[\mu_{3-k}f_1(\sigma)-f_2(\sigma)\right]\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{\mathrm{d}\sigma}{\sigma}\right\}. \tag{29}$$

From Savin (1961), the first integral in Eq. (29) is:

$$\oint_\gamma\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{\log(\sigma)}{\sigma}\,\mathrm{d}\sigma=4\pi\mathrm{i}\log(\sigma_c-\zeta)+2\pi^2, \tag{30}$$

where σ_c is defined as $\sigma_c=\mathrm{e}^{\mathrm{i}\theta_c}$.

An expression for the stress functions $\{\Phi_k^0(\zeta)\}_{k=1,2}$ can now be derived by substituting Eqs. (27), (28) and (30) into Eq. (29). Then the stress functions $\{\Phi_k(\zeta)\}_{k=1,2}$ are obtained by substituting $\{\Phi_k^0(\zeta)\}_{k=1,2}$ into Eq. (12). The calculations are straightforward but lengthy. Only the final results are presented here:

$$\Phi_k(\zeta,\theta_c)=-F\left\{\Delta_k\log(\zeta)-\frac{(-1)^k\mathrm{i}}{2\pi(\mu_1-\mu_2)}\log\left(1-\zeta\mathrm{e}^{i\theta_c}\right)\right\}-\frac{(-1)^k\mathrm{i}R}{4\pi(\mu_1-\mu_2)}J_k(\zeta,\theta_c) \tag{31a}$$

with

$$J_k(\zeta)=\frac{1}{2}\int_{\sigma_c}^{1/\sigma_c}\left[U_k(\sigma)-U_k(\sigma_c)\right]\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{\mathrm{d}\sigma}{\sigma}, \tag{31b}$$

$$U_k(\sigma)=\int p(\sigma)\left[-\mu_{3-k}\left(1-\frac{1}{\sigma^2}\right)+\mathrm{i}\left(1+\frac{1}{\sigma^2}\right)\right]\mathrm{d}\sigma - \int t(\sigma)\left[\mathrm{i}\mu_{3-k}\left(1+\frac{1}{\sigma^2}\right)+\left(1-\frac{1}{\sigma^2}\right)\right]\mathrm{d}\sigma. \tag{31c}$$

The stress functions $\phi_k(z_k)=\Phi_k(\zeta(z_k))$ could then be directly obtained from Eq. (31) by changing ζ into z_k using Eq. (11).

The next step is to derive a closed form expression for the stress functions $\phi_k(z_k)$ under the form of a series with unknown coefficients. For this purpose, the stress distributions $p(\theta)$ and $t(\theta)$ in Eq. (31c) are expressed as a Fourier series with unknown coefficients. These series are

$$\begin{cases}p(\theta)=P_0+\sum_1^\infty P_n\cos(n\theta),\\ t(\theta)=\sum_1^\infty T_n\sin(n\theta).\end{cases} \tag{32}$$

The coefficients P_0 and $\{P_n,T_n\}_{k=1\ldots n}$ are unknown.

Using the relation $\sigma = e^{i\theta}$, $p(\sigma)$ and $t(\sigma)$ take the following form:

$$p(\sigma) = P_0 + \frac{1}{2} P_n \sum_1^\infty (\sigma^n + \sigma^{-n}), \qquad t(\sigma) = -\frac{i}{2} T_n \sum_1^\infty (\sigma^n - \sigma^{-n}). \tag{33}$$

The functions $U_k(\sigma)$, Eq. (31c) can then be calculated by substituting Eq. (33) into Eq. (31c) and integrating. They are

$$U_k(\sigma) = \frac{1}{2} \bar{P}_2 \mu_{3-k} \left(\sigma + \frac{1}{\sigma}\right) + \frac{i}{2} \bar{T}_2 \left(\sigma - \frac{1}{\sigma}\right) + i \bar{T}_1 \log(\sigma) + \frac{1}{2} \sum_{n=1}^{\infty} \bar{P}_{n+2} \frac{\mu_{3-k}}{n+1} \left(\sigma^{n+1} + \frac{1}{\sigma^{n+1}}\right)$$
$$+ \frac{1}{2} \sum_{n=1}^{\infty} \bar{T}_{n+2} \frac{i}{n+1} \left(\sigma^{n+1} - \frac{1}{\sigma^{n+1}}\right). \tag{34}$$

The new coefficients $\{\bar{P}_n, \bar{T}_n\}$ are related to the coefficients $\{P_n, T_n\}$ as follows:

$$\begin{cases} \bar{P}_2 = P_2 - T_2 - 2P_0, \\ \bar{P}_n = P_n - T_n - P_{n-2} - T_{n-2} \ (n \geqslant 3) \end{cases} \quad \text{and} \quad \begin{cases} \bar{T}_1 = P_1 - T_1, \\ \bar{T}_2 = P_2 - T_2 + 2P_0, \\ \bar{T}_n = P_n - T_n + P_{n-2} + T_{n-2} \quad (n \geqslant 3). \end{cases} \tag{35}$$

The new expression (32) for $U_k(\sigma)$ may now be substituted into Eq. (31b) in order to calculate $J_k(\zeta)$. This requires the calculation of the following integrals:

$$\int_{\sigma_c}^{1/\sigma_c} \log(\sigma) \left(\frac{\sigma + \zeta}{\sigma - \zeta}\right) \frac{d\sigma}{\sigma} \quad \text{and} \quad \int_{\sigma_c}^{1/\sigma_c} \left(\frac{\sigma + \zeta}{\sigma - \zeta}\right) \sigma^{\pm n} \, d\sigma, \quad n \in \mathbb{N}. \tag{36}$$

Let us define the second integral in Eq. (36) as

$$E_{\pm n}(\zeta) = \int_{\sigma_c}^{1/\sigma_c} \left(\frac{\sigma + \zeta}{\sigma - \zeta}\right) \sigma^{\pm n} \, d\sigma. \tag{37}$$

From Gradshteyn and Rydzhik (1980), these integrals are

$$E_0(\zeta) = [\sigma + 2\zeta \log(\sigma - \zeta)]_{\sigma_c}^{1/\sigma_c} = \frac{1}{\sigma_c} - \sigma_c + 2\zeta \log\left(\frac{1 - \zeta\sigma_c}{\sigma_c - \zeta}\right) - 2\zeta \log(\sigma_c), \tag{38}$$

and

$$E_{-1}(\zeta) = [2 \log(\sigma - \zeta) - \log(\sigma)]_{\sigma_c}^{1/\sigma_c} = 2 \log\left(\frac{1 - \zeta\sigma_c}{\sigma_c - \zeta}\right), \tag{39}$$

and

$$E_n(\zeta) = \left[\frac{\sigma^{n+1}}{n+1} + 2\zeta^{n+1} \log(\sigma - \zeta) + 2\zeta \sum_{p=0}^{n-1} \frac{\zeta^p \sigma^{n-p}}{n-p}\right]_{\sigma_c}^{1/\sigma_c} \tag{40a}$$

$$\Rightarrow \quad E_n(\zeta) = \frac{1}{n+1} \left[\frac{1}{\sigma_c^{n+1}} - \sigma_c^{n+1}\right] + 2\zeta^{n+1} \log\left(\frac{1 - \zeta\sigma_c}{\sigma_c - \zeta}\right)$$
$$- 2\zeta^{n+1} \log(\sigma_c) + 2\zeta \sum_{p=0}^{n-1} \frac{\zeta^p}{n-p} \left[\frac{1}{\sigma_c^{n-p}} - \sigma_c^{n-p}\right] \quad \text{for } n \geqslant 1, \tag{40b}$$

and

$$E_{-n}(\zeta) = \left[\frac{-1}{(n-1)\,\sigma^{n-1}} + \frac{2}{\zeta^{n-1}}\,[\log(\sigma-\zeta) - \log(\sigma)] + 2\zeta \sum_{p=1}^{n-1} \frac{1}{(n-p)\,\zeta^p \sigma^{n-p}} \right]_{\sigma_c}^{1/\sigma_c} \tag{41a}$$

$$\Rightarrow E_{-n}(\zeta) = \frac{1}{n-1}\left[\frac{1}{\sigma_c^{n-1}} - \sigma_c^{n-1}\right] + \frac{2}{\zeta^{n-1}}\log\left(\frac{1-\zeta\sigma_c}{\sigma_c-\zeta}\right) + \frac{2}{\zeta^{n-1}}\log(\sigma_c) + 2\zeta\sum_{p=1}^{n-1}\frac{1}{(n-p)\,\zeta^p}\left[\sigma_c^{n-p} - \frac{1}{\sigma_c^{n-p}}\right] \quad \text{for} \quad n \geqslant 2. \tag{41b}$$

Calculating the first integral in Eq. (36), the result is

$$\int_{\sigma_c}^{1/\sigma_c} \log(\sigma)\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{d\sigma}{\sigma} = 2\log(\zeta)\log\left(\frac{1-\zeta\sigma_c}{(\sigma_c-\zeta)\,\sigma_c}\right) + 2\left[F^*\left(\frac{1}{\zeta\sigma_c}\right) - F^*\left(\frac{\sigma_c}{\zeta}\right)\right], \tag{42}$$

where the function $F\log(z)$ is well-behaved and analytic except on $]-\infty, 0]$, defined as

$$F^*(z) = \int_0^z \frac{\log(\sigma)}{\sigma-1}\,d\sigma \tag{43}$$

Details of the calculations leading to Eq. (42) are given in Appendix B.

The functions $J_k(\zeta)$ in Eq. (31b) are determined by substituting $U_k(\zeta)$, Eq. (34), into Eq. (31b) and by using Eqs. (38)–(42):

$$J_k(\zeta,\theta_c) = \frac{1}{4}\bar{P}_2\,\mu_{3-k}\,(E_0(\zeta) + E_{-2}(\zeta)) + \frac{i}{4}\bar{T}_2\,(E_0(\zeta) - E_{-2}(\zeta)) - \frac{1}{2}\,U_k(\theta_c)E_{-1}(\zeta) + \frac{1}{4}\sum_1^{\infty}\frac{\mu_{3-k}}{n+1}\bar{P}_{n+2}\left[E_n(\zeta) + E_{-(n+2)}(\zeta)\right] + \frac{1}{4}\sum_1^{\infty}\frac{i}{n+1}\bar{T}_{n+2}\left[E_n(\zeta) - E_{-(n+2)}(\zeta)\right] + i\bar{T}_1\left\{\log(\zeta)\left[\log\left(\frac{1-e^{i\theta_c}\zeta}{e^{i\theta_c}-\zeta}\right) - i\theta_c\right] + F\log\left(\frac{1}{e^{i\theta_c}\zeta}\right) - F\log\left(\frac{e^{i\theta_c}}{\zeta}\right)\right\}. \tag{44}$$

The function $U_k(\theta_c)$ in Eq. (44) is defined as $U_k(\theta_c) = U_k(\sigma_c = e^{i\theta_c})$ with $U_k(\sigma)$ given by Eq. (34). $U_k(\theta_c)$ is obtained the following form:

$$U_k(\theta_c) = -\bar{T}_1\,\theta_c + \sum_0^{\infty}\left\{\mu_{3-k}\bar{P}_{n+2}\,\frac{\cos[(n+1)\theta_c]}{n+1} - \bar{T}_{n+2}\,\frac{\sin[(n+1)\theta_c]}{n+1}\right\}. \tag{45}$$

The resultant force F given by Eq. (26) can also be expressed as a function of $\{\bar{P}_n, \bar{T}_n\}$ by replacing $p(\theta)$ and $t(\theta)$ by their Fourier series expressions (32), and integrating over θ:

$$F(\theta_c) = -R\left\{\bar{T}_1\,\theta_c + \sum_{n=2}^{\infty}\bar{T}_n\frac{\sin[(n-1)\theta_c]}{n-1}\right\}. \tag{46}$$

Now, substituting F from Eq. (46), and $J_k(\zeta,\theta_c)$ from Eqs. (44) and (45) into Eq. (31a), we can write the stress functions $\phi_k(z_k,\theta_c) = \Phi_k(\zeta,\theta_c)$ as an infinite sum of known functions of ζ and θ_c multiplied by the coefficients $\{\bar{P}_n, \bar{T}_n\}$. These functions can be expressed in terms of z_k and θ_c by using Eq. (11). The stress functions, after straightforward but extensive calculations are

$$\phi_k(z_k,\theta_c) = \bar{T}_1\,\Omega_1^{(T)}(z_k,k,\theta_c) + \sum_{n=2}^{\infty}\frac{\bar{T}_n}{n-1}\,\Omega_n^{(T)}(z_k,k,\theta_c) + \sum_{n=2}^{\infty}\frac{\bar{P}_n}{n-1}\,\Omega_n^{(P)}(z_k,k,\theta_c), \quad k = 1,2. \tag{47}$$

The functions $\Omega_1^{(T)}(z_k,k,\theta_c)$, $\{\Omega_n^{(T)}(z_k,k,\theta_c)\}_{n\geqslant 2}$, and $\{\Omega_n^{(P)}(z_k,k,\theta_c)\}_{n\geqslant 2}$ are given in Appendix C.

Substituting the stress function Equations (47) into Eqs. (1) and (3), the displacements (u,v) and the stresses $(\sigma_x, \sigma_y, \tau_{xy})$ can be written as infinite series in terms of coefficients $\{\bar{P}_n, \bar{T}_n\}$. Similarly, the stresses in polar coordinates $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ can be obtained as infinite series with coefficients $\{\bar{P}_n, \bar{T}_n\}$ by substituting the stress functions Eq. (47) into Eq. (48a)–(48c):

$$\sigma_r = 2\,\Re\mathrm{e}\left\{(\sin(\theta) - \mu_1\cos(\theta))^2\phi_1'(z_1,\theta_c) + (\sin(\theta) - \mu_2\cos(\theta))^2\phi_2'(z_2,\theta_c)\right\}, \tag{48a}$$

$$\sigma_\theta = 2\,\Re\mathrm{e}\left\{(\mu_1\sin(\theta) + \cos(\theta))^2\phi_1'(z_1,\theta_c) + (\mu_2\sin(\theta) + \cos(\theta))^2\phi_2'(z_2,\theta_c)\right\}, \tag{48b}$$

$$\tau_{r\theta} = -2\,\Re\mathrm{e}\left\{\begin{array}{l}(\sin(\theta) - \mu_1\cos(\theta))(\mu_1\sin(\theta) + \cos(\theta))\phi_1'(z_1,\theta_c)\\ +(\sin(\theta) - \mu_2\cos(\theta))(\mu_2\sin(\theta) + \cos(\theta))\phi_2'(z_2,\theta_c)\end{array}\right\}. \tag{48c}$$

2.3. Verification

The stress functions $\phi_k(z_k,\theta_c)$ given by Eq. (47) were specifically developed so that they would give zero stresses in the region of the hole boundary defined by $|\theta| > \theta_c$. This can be verified by arbitrarily choosing an angle θ_c and coefficients P_0 and $\{P_n, T_n\}_{n\geqslant 1}$, and then by calculating the resulting stress functions and stresses throughout the plate. The stresses $\sigma_r(R,\theta)$ and $\tau_{r\theta}(R,\theta)$ around the hole should be equal to the normal stress $p(\theta)$ and shear stress $t(\theta)$ on the hole boundary (corresponding to the chosen coefficients $\{P_n, T_n\}$) for $|\theta| \leqslant \theta_c$ and be identically equal to zero for $|\theta| > \theta_c$.

A typical composite laminate made of T800/924C with the layup $[(45/-45/(0)_3]_s$ was chosen to verify the validity of Eq. (47). The elastic properties of that laminate were $E_x = 105.29$ GPa, $E_y = 23.64$ GPa, $G_{xy} = 20.04$ GPa, $v_{xy} = 0.67$. The contact angle θ_c was chosen to be equal to 60°. The normal stress distribution $p(\theta)$ and tangential stress distribution $t(\theta)$ over [0,60] were arbitrarily taken as

$$p(\theta) = 1 - 3\cos(\theta) + 5\cos(2\theta) - 4\cos(3\theta), \tag{49a}$$

$$t(\theta) = 2\sin(\theta) - 4\sin(2\theta) + 3\sin(3\theta). \tag{49b}$$

The coefficients $\{P_n, T_n\}_{k=1\ldots n}$, Eq. (32), are then known.

Fig. 4(a) shows the chosen normal stress $p(\theta)$ and the calculated normal stress $\sigma_r(R,\theta)$ over the [0,90] interval. Fig. 4(b) shows the chosen tangential stress $t(\theta)$ and the calculated tangential stress $\tau_{r\theta}(R,\theta)$.

The calculated stresses $\sigma_r(R,\theta)$ and $\tau_{r\theta}(R,\theta)$ match very well with the normal and shear stresses $p(\theta)$ and $t(\theta)$, respectively up to $\theta = \theta_c$, and the stresses $\sigma_r(R,\theta)$ and $\tau_{r\theta}(R,\theta)$ are equal to zero for θ greater than θ_c. This proves that the stress-free boundary conditions are indeed identically satisfied by the newly derived stress functions given by Eq. (47).

3. Applications

Because the stress functions $\{\phi_k\}_{k=1,2}$ automatically satisfy the no-stress boundary condition along the hole for θ greater than θ_c, the remaining boundary conditions Eq. (20) need only be applied on the $[0,\theta_c]$ interval instead of on the whole interval [0,90]. Very often, such as for instance, in pin-loaded holes problems, the numerical method of collocation is used to enforce the boundary conditions. Using the above stress functions should consequently lead to a less time-consuming collocation method, since less collocation points will be required.

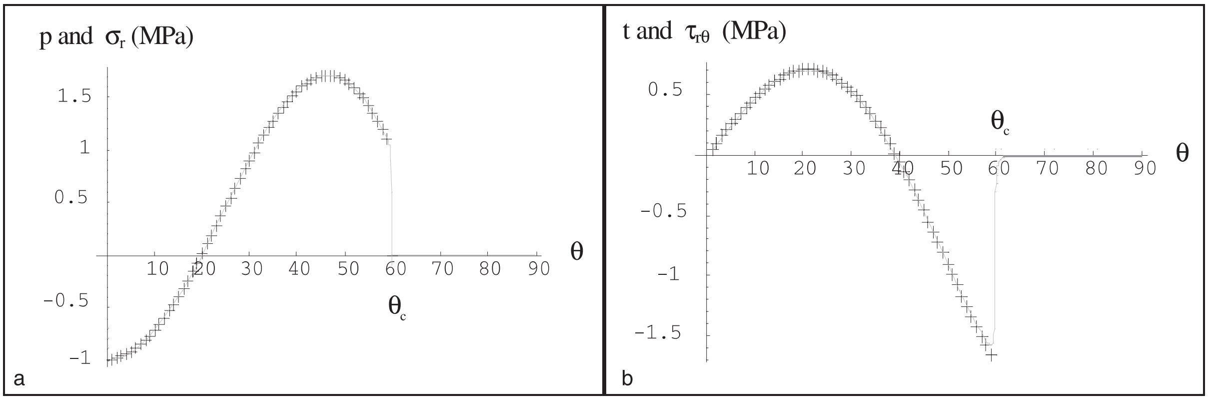

Fig. 4. (a) Stress distribution $p(\theta)$ (+) and normal stress $\sigma_r(R,\theta)$ (—) on the hole boundary for a contact angle $\theta_c = 60°$ and for $p(\theta) = 1 - 3\cos(\theta) + 5\cos(2\theta) - 4\cos(3\theta)$, (b) Stress distribution $t(\theta)$ (+) and tangential stress $\tau_{r\theta}(R,\theta)$ (—) on the hole boundary for a contact angle $\theta_c = 60°$ and for $t(\theta) = 2\sin(\theta) - 4\sin(2\theta) + 3\sin(3\theta)$.

In the case of a pin-loaded hole with a pin-hole clearance, the part of the hole where the pin is in contact with the plate $[-\theta_c,\theta_c]$ is subjected to displacement boundary conditions due to the pin, or displacement/stress boundary conditions in the case of friction. The rest of the hole boundary is stress-free, since the pin is not in contact with the hole (Fig. 5).

4. Conclusions

A new approach has been presented to address problems in composite plates with a hole, where a part of the hole boundary is stress-free. Stress functions were derived, which automatically give normal and tan-

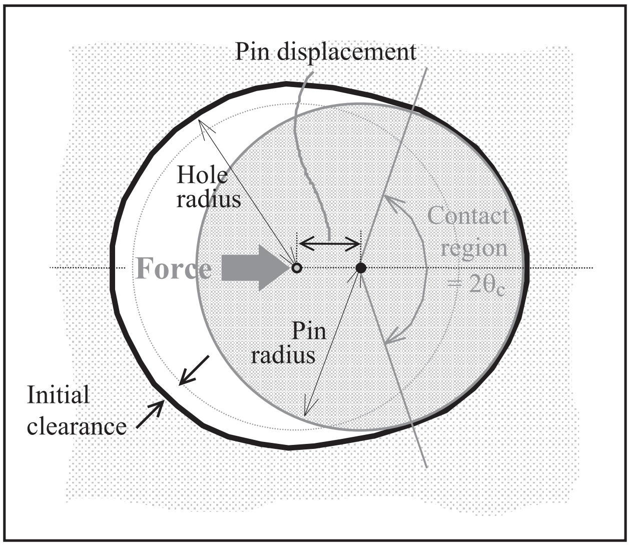

Fig. 5. Pin and hole geometry.

gential stresses equal to zero over the unloaded part of the hole boundary. It was verified numerically that these stress functions indeed possessed that property. The present approach is quite general and could be applied to a range of problems. In a forthcoming article, it will be applied to the case of a pin-loaded hole with a pin-hole clearance.

The method could also be extended to include the case, where the composite plate is subjected to far-field boundary conditions. Using the superposition principle (Ireman et al., 1993), stress functions for a plate with a hole under multidirectional remote loading could be added to the above stress functions, (Eq. (47)). This will yield stresses and displacements for a composite plate with a hole subjected to loading both on the hole boundary and at infinity (multiaxial loading).

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Appendix A

The coefficients μ_1 and μ_2 are the complex roots (two by two conjugate: $\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2$) of the characteristic equation:

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0.$$

In the case of an orthotropic plate, i.e. when the load is along the principal axes of the plate, this characteristic equation becomes

$$\mu^4 + \left(\frac{E_{xx}}{G_{xy}} - 2\,v_{xy}\right)\mu^2 + \frac{E_{xx}}{E_{yy}} = 0.$$

The roots can then be written in closed form, and are

$$\underset{k=1,2}{\mu_k} = \begin{cases} \frac{(-1)^{k-1}}{\sqrt{2}}\sqrt{\sqrt{\frac{E_{xx}}{E_{yy}}} - \left(\frac{E_{xx}}{2G_{xy}} - v_{xy}\right)} + \frac{\mathrm{i}}{\sqrt{2}}\sqrt{\sqrt{\frac{E_{xx}}{E_{yy}}} + \left(\frac{E_{xx}}{2G_{xy}} - v_{xy}\right)} & \text{if } \sqrt{\frac{E_{xx}}{E_{yy}}} > \frac{E_{xx}}{2G_{xy}} - v_{xy}, \\ \frac{\mathrm{i}}{\sqrt{2}}\left((-1)^{k-1}\sqrt{\left(\frac{E_{xx}}{2G_{xy}} - v_{xy}\right) - \sqrt{\frac{E_{xx}}{E_{yy}}}} + \sqrt{\sqrt{\frac{E_{xx}}{E_{yy}}} + \left(\frac{E_{xx}}{2G_{xy}} - v_{xy}\right)}\right) & \text{if } \sqrt{\frac{E_{xx}}{E_{yy}}} < \frac{E_{xx}}{2G_{xy}} - v_{xy}. \end{cases}$$

The roots μ_k are written in complex form as $\mu_k = \alpha_k + \mathrm{i}\beta_k$, which defines $\{\alpha_k, \beta_k\}_{k=1,2}$.

The coefficients p_1, q_1, p_2, q_2 appearing in Eq. (1) are given by:

$$\begin{cases} p_k = a_{11}\,\mu_k^2 + a_{12} - a_{16}\,\mu_k, \\ q_k = a_{12}\,\mu_k + \frac{a_{22}}{\mu_k} - a_{26}, \end{cases}$$

for $k = 1, 2$. The coefficients a_{11}, a_{22}, and a_{12} are the compliances in plane stress or plane strain.

The complex constants $\{\Delta_k\}_{k=1,2}$ in Eq. (12) have the following real and imaginary parts:

$$\Re e\{\Delta_k\} = \frac{\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_2^2+\beta_2^2\right)\left\{\begin{array}{l} a_{12}\left[(\alpha_1-\alpha_2)^2+\beta_{3-k}^2-\beta_k^2\right] \\ +a_{22}\left[\frac{4\alpha_{3-k}^2-\alpha_k^2+\beta_k^2}{\alpha_{3-k}^2+\beta_{3-k}^2}-\frac{2\alpha_k\alpha_{3-k}}{\alpha_k^2+\beta_k^2}-1\right]\end{array}\right\}}{4\pi a_{22}\beta_k\left[(\alpha_1-\alpha_2)^4+2(\alpha_1-\alpha_2)^2\left(\beta_1^2+\beta_2^2\right)+\left(\beta_1^2-\beta_2^2\right)^2\right]},$$

$$\Im m\{\Delta_k\} = (-1)^{k+1}\frac{\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_2^2+\beta_2^2\right)\left\{a_{12}(\alpha_2-\alpha_1)+a_{22}\left[\frac{\alpha_1}{\alpha_2^2+\beta_2^2}-\frac{\alpha_2}{\alpha_1^2+\beta_1^2}\right]\right\}}{2\pi a_{22}\left[(\alpha_1-\alpha_2)^4+2(\alpha_1-\alpha_2)^2\left(\beta_1^2+\beta_2^2\right)+\left(\beta_1^2-\beta_2^2\right)^2\right]}.$$

Appendix B

The integral in Eq. (40) is:

$$\int_{\sigma_c}^{1/\sigma_c}\log(\sigma)\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{d\sigma}{\sigma} = 2\int_{\sigma_c}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma-\zeta}d\sigma-\int_{\sigma_c}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma}d\sigma$$

$$\Rightarrow\quad \int_{\sigma_c}^{1/\sigma_c}\log(\sigma)\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{d\sigma}{\sigma} = 2\int_{\sigma_c}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma-\zeta}d\sigma-\left[\frac{\log^2(\sigma)}{2}\right]_{\sigma_c}^{1/\sigma_c} = 2\int_{\sigma_c}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma-\zeta}d\sigma.$$

This last integral cannot be expressed in terms of elementary functions. However, by performing the change of variable $\sigma\to\zeta\sigma$, this integral can be written as

$$\int_{\sigma_c}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma-\zeta}d\sigma = \log(\zeta)\int_{\sigma_c/\zeta}^{1/\zeta\sigma_c}\frac{d\sigma}{\sigma-1}+\int_{\sigma_c/\zeta}^{1/\zeta\sigma_c}\frac{\log(\sigma)}{\sigma-1}d\sigma$$

$$\Rightarrow\quad \int_{\sigma_c}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma-\zeta}d\sigma = \log(\zeta)\left[\log\left(\frac{1-\zeta\sigma_c}{\sigma_c-\zeta}\right)-\log(\sigma_c)\right]+\int_{\sigma_c/\zeta}^{1/\sigma_c}\frac{\log(\sigma)}{\sigma-1}d\sigma$$

$$\int_{\sigma_c}^{1/\sigma_c}\log(\sigma)\left(\frac{\sigma+\zeta}{\sigma-\zeta}\right)\frac{d\sigma}{\sigma} = 2\log(\zeta)\log\left(\frac{1-\zeta\sigma_c}{(\sigma_c-\zeta)\sigma_c}\right)+2\left[F^*\left(\frac{1}{\zeta\sigma_c}\right)-F^*\left(\frac{\sigma_c}{\zeta}\right)\right],$$

where the function $F^*(z)$ is defined as follows

$$F^*(z) = \int_0^z\frac{\log(\sigma)}{\sigma-1}d\sigma.$$

The function $F\log(z)$ is well behaved and analytic except on $]-\infty,0]$. It is continuous as $\lim_{\sigma\to 1}\{\log(\sigma)/(\sigma-1)\}=1$ and it takes a finite value at 0 as

$$F^*(1) = \int_0^1\frac{\log(\sigma)}{\sigma-1}d\sigma = \frac{\pi^2}{6}.$$

Appendix C

The functions $\Omega_1^{(T)}\ (z_k,\ k,\ \theta_c)$, $\{\Omega_n^{(T)}(z_k,k,\theta_c)\}_{n\geqslant 2}$, and $\{\Omega_n^{(P)}(z_k,k,\theta_c)\}_{n\geqslant 2}$ in Eq. (47) are

$$\Omega_1^{(T)}(z_k,k,\theta_c) = R\theta_c\left\{\begin{array}{l}\Delta_k \log\left[\frac{z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}}{R(1+\mathrm{i}\mu_k)}\right]\\ +\frac{\mathrm{i}\,(-1)^k}{2\pi(\mu_1-\mu_2)}\log\left[1-\mathrm{e}^{\mathrm{i}\theta_c}\frac{z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}}{R(1+\mathrm{i}\mu_k)}\right]\end{array}\right\} + \frac{R(-1)^k}{4\pi(\mu_1-\mu_2)}$$
$$\times\left\{\begin{array}{l}\log\left[\frac{z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}}{R(1+\mathrm{i}\mu_k)}\right]\left\{\log\left[\frac{R(1+\mathrm{i}\mu_k)-\mathrm{e}^{\mathrm{i}\theta_c}\left(z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}\right)}{R(1+\mathrm{i}\mu_k)\,\mathrm{e}^{\mathrm{i}\theta_c}-\left(z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}\right)}\right]-\mathrm{i}\theta_c\right\}\\ -\frac{\mathrm{i}\theta_c}{2}E_{-1}(z_k)+\left[F^*\left(\frac{1}{\zeta\,\mathrm{e}^{\mathrm{i}\theta_c}}\right)-F^*\left(\frac{\mathrm{e}^{\mathrm{i}\theta_c}}{\zeta}\right)\right]\end{array}\right\},$$

$$\Omega_n^{(T)}(z_k,k,\theta_c) = \frac{R(-1)^k}{16\pi(\mu_1-\mu_2)}\{E_{n-2}(z_k)-E_{-n}(z_k)-2\mathrm{i}E_{-1}(z_k)\sin[(n-1)\theta_c]\}$$
$$+R\sin[(n-1)\theta_c]\left\{\begin{array}{l}\Delta_k \log\left[\frac{z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}}{R(1+\mathrm{i}\mu_k)}\right]\\ +\frac{\mathrm{i}\,(-1)^k}{2\pi(\mu_1-\mu_2)}\log\left[1-\mathrm{e}^{\mathrm{i}\theta_c}\frac{z_k-\sqrt{z_k^2-R^2\left(1+\mu_k^2\right)}}{R(1+\mathrm{i}\mu_k)}\right]\end{array}\right\},\quad n\geqslant 2,$$

$$\Omega_n^{(P)}(z_k,k,\theta_c) = -\mathrm{i}\,\frac{R(-1)^k\mu_{3-k}}{16\pi(\mu_1-\mu_2)}\{E_{n-2}(z_k)+E_{-n}(z_k)-2E_{-1}(z_k)\cos[(n-1)\theta_c]\},\quad n\geqslant 2.$$

The functions $\{E_{\pm n}(\zeta)\}$ appear above as $\{E_{n-2}(z_k)\pm E_{-n}(z_k)\}_{n\geqslant 2}$ and $E_{-1}(z_k)$. From Eqs. (38)–(41), expressions for $\{E_{n-2}(z_k)\pm E_{-n}(z_k)\}_{n\geqslant 2}$ and $E_{-1}(z_k)$ can be derived by using Eq. (11), which relate ζ to z_k. The expressions for $\{E_{n-2}(z_k)\pm E_{-n}(z_k)\}_{n\geqslant 2}$ and $E_{-1}(z_k)$ are

$$E_{-1}(z_k) = 2\log\left(\frac{R(1+i\mu_k)-\mathrm{e}^{\mathrm{i}\theta_c}\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}{R(1+\mathrm{i}\mu_k)\,\mathrm{e}^{\mathrm{i}\theta_c}-\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}\right),$$

$$E_0(z_k)+E_{-2}(z_k) = \frac{4\mathrm{i}\,\theta_c}{R}\left[\frac{\mathrm{i}\mu_k z_k+\sqrt{z_k^2-R^2(1+\mu_k^2)}}{1+\mu_k^2}\right]$$
$$+\frac{4}{R}\left[\frac{\mathrm{i}\mu_k\sqrt{z_k^2-R^2(1+\mu_k^2)}+z_k}{1+\mu_k^2}\right]\log\left(\frac{R(1+\mathrm{i}\mu_k)-\mathrm{e}^{\mathrm{i}\theta_c}\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}{R(1+\mathrm{i}\mu_k)\,\mathrm{e}^{\mathrm{i}\theta_c}-\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}\right),$$

$$E_0(z_k)-E_{-2}(z_k) = -\frac{4\mathrm{i}\,\theta_c}{R}\left[\frac{\mathrm{i}\mu_k\sqrt{z_k^2-R^2(1+\mu_k^2)}+z_k}{1+\mu_k^2}\right]-4\mathrm{i}\sin(\theta_c)$$
$$-\frac{4}{R}\left[\frac{\mathrm{i}\mu_k z_k+\sqrt{z_k^2-R^2(1+\mu_k^2)}}{1+\mu_k^2}\right]\log\left(\frac{R(1+\mathrm{i}\mu_k)-\mathrm{e}^{\mathrm{i}\theta_c}\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}{R(1+i\mu_k)\,\mathrm{e}^{\mathrm{i}\theta_c}-\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}\right),$$

$$
\begin{aligned}
E_{n-2}(z_k) + E_{-n}(z_k) = {} & 4\,\theta_c \sin\left[-\mathrm{i}(n-1)\log\left(\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right)\right] \\
& + 4\cos\left[-\mathrm{i}(n-1)\log\left(\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right)\right] \\
& \times \log\left(\frac{R(1+\mathrm{i}\mu_k)-\mathrm{e}^{\mathrm{i}\theta_c}\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}{R(1+\mathrm{i}\mu_k)\,\mathrm{e}^{\mathrm{i}\theta_c}-\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}\right) \\
& + 8\sum_{p=0}^{n-3}\frac{\sin[(n-2-p)\theta_c]}{(n-2-p)}\sin\left[-\mathrm{i}(p+1)\log\left(\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right)\right] \quad \text{for } n \geqslant 3.
\end{aligned}
$$

$$
\begin{aligned}
E_{n-2}(z_k) - E_{-n-2}(z_k) = {} & -4\mathrm{i}\theta_c \cos\left[-\mathrm{i}(n-1)\log\left(\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right)\right] - \frac{4\mathrm{i}}{n-1}\sin[(n-1)\theta_c] \\
& + 4i\sin\left[-\mathrm{i}(n-1)\log\left(\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right)\right] \\
& \times \log\left(\frac{R(1+\mathrm{i}\mu_k)-\mathrm{e}^{\mathrm{i}\theta_c}\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}{R(1+\mathrm{i}\mu_k)\,\mathrm{e}^{\mathrm{i}\theta_c}-\left(z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}\right)}\right) \\
& - 8\mathrm{i}\sum_{p=0}^{n-3}\frac{\sin[(n-2-p)\theta_c]}{(n-2-p)}\cos\left[-\mathrm{i}(p+1)\log\left(\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right)\right] \quad \text{for } n \geqslant 3.
\end{aligned}
$$

The expression for the stress functions $\phi_k(z_k)$ can be simplified if they are assumed to be independent of the angle θ_c (but do not satisfy the stress-free boundary conditions):

$$
\phi_k(z_k) = \bar{T}_1\,\Omega_1^{(T)}(z_k,k) + \sum_{n=2}^{\infty}\frac{\bar{T}_n}{n-1}\,\Omega_n^{(T)}(z_k,k) + \sum_{n=2}^{\infty}\frac{\bar{P}_n}{n-1}\,\Omega_n^{(P)}(z_k,k).
$$

The stress functions have a form similar to Eq. (47), but the functions $\Omega_1^{(T)}(z_k,k)$, $\{\Omega_n^{(T)}(z_k,k)\}n \geqslant 2$, and $\{\Omega_n^{(P)}(z_k,k)\}n \geqslant 2$ are considerably simpler than the functions $\Omega_1^{(T)}(z_k,k,\theta_c)$, $\{\Omega_n^{(T)}(z_k,k,\theta_c)\}n \geqslant 2$, and $\{\Omega_n^{(P)}(z_k,k,\theta_c)\}n \geqslant 2$:

$$
\Omega_1^{(T)}(z_k,k) = \pi R\,\Delta_k \log\left[\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right],
$$

$$
\Omega_n^{(T)}(z_k,k) = \frac{-\mathrm{i}(-1)^k R}{4(\mu_1-\mu_2)}\left[\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right]^{n-1} \quad \text{for } n \geqslant 2,
$$

$$
\Omega_n^{(T)}(z_k,k) = \frac{-(-1)^k R\,\mu_{3-k}}{4(\mu_1-\mu_2)}\left[\frac{z_k-\sqrt{z_k^2-R^2(1+\mu_k^2)}}{R(1+\mathrm{i}\mu_k)}\right]^{n-1} \quad \text{for } n \geqslant 2.
$$

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